MATH 135 Lecture Notes Section 3.2 Slope of a Tangent Line
September 14, 2006

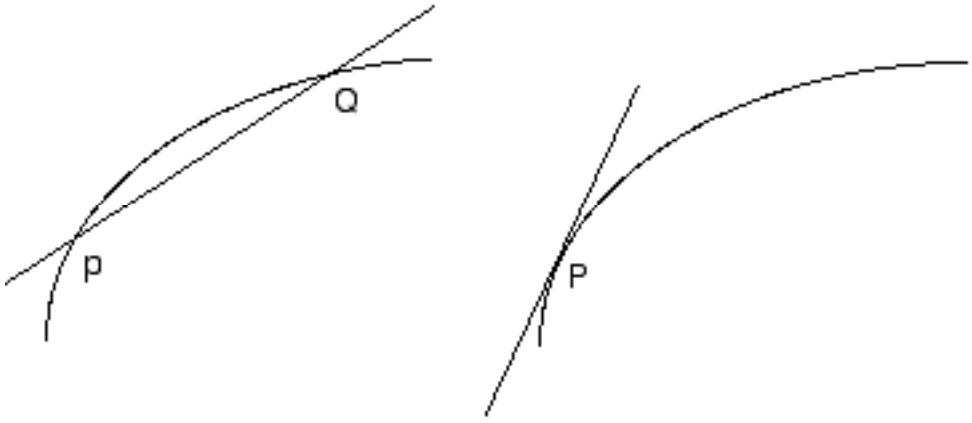
Review definition of **slope**: The slope of a line is defined as the ratio of the change in *y*-values (rise) to the corresponding change in *x*-values (run) of the coordinates of any two points on the line. Therefore. the slope *m* is given by (page 29).

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

The quantity Δx is not to be thought of as Δ times x for the symbol Δ used here has no meaning by itself. The name **increment** is given to the difference of the coordinates of two points, and therefore Δx and Δy are the increments in x and y respectively (page 78).

The slope of a line is also the **rate of change** of the dependent variable (y) with respect to the independent variable (x). (from elementary algebra)

Straight lines have constant slopes. Most curves, however, do not. We may find the slope of a line passing through two points on the curve, which is called a **secant** line. Or we may find the slope of a line that touches the curve at one point. This line is called a **tangent line**.* The figure at the left shows a secant to the curve through points P and Q. The figure at the right shows a tangent to the curve at point P.



* This informal description is not entirely accurate, but will do for now.

We may find the slope of a secant line by using the above formula for slope with the coordinates for P(x_1, y_1) and Q(x_2, y_2). In Example 1, page 78, this is done for P(2, 10) and five different points Q: Q_1, Q_2, Q_3, Q_4, and Q_5. Each of the slopes is the slope of a secant line. However, as point Q approaches point P, the slopes of the secant lines appear to be approaching the number 7. Therefore the slope of the tangent line at (2, 10) is 7.

Example 2, page 79, shows how this slope may be calculated using the increment method. (I will give a slightly different derivation from the one in the book). The coordinates of point P are (2, 10). The coordinates of point Q are $(2 + \Delta x\ ,\ 10 + \Delta y) = (2 + \Delta x\ ,\ f(2 + \Delta x))$. We need an expression for
$f(2 + \Delta x))$. so we substitute $2 + \Delta x$ for x in $f(x) = x^2 + 3x$.

$$f(2 + \Delta x)) = (2 + \Delta x))^2 + 3(2 + \Delta x) = 4 + 4\Delta x + (\Delta x)^2 + 6x + 3\Delta x = 10 + 7\Delta x + (\Delta x)^2.$$

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \frac{10 + 7\Delta x + (\Delta x)^2 - 10}{\Delta x} = \frac{7\Delta x + (\Delta x)^2}{\Delta x} = \frac{\Delta x(7 + \Delta x)}{\Delta x} = 7 + \Delta x$$

$$m_{\tan} = \lim_{\Delta x \to 0} m_{pq} = \lim_{\Delta x \to 0} (7 + \Delta x) = 7$$

We see that this result agrees with that found in Example 1.

Examples 3 and 4 (page 80) show how to find the slope of a line tangent to a curve at a general point (x_1, y_1). The points P and Q are $P(x_1, y_1)$ and $Q(x_1 + \Delta x, y_1 + \Delta y)$. Another way to represent the coordinates is $P(x_1, f(x_1))$ and $Q(x_1 + \Delta x, f(x_1 + \Delta x))$. This chart shows the relationship among the various ways of showing the coordinates of P and Q.

x-coordinate of P	x_1	x_1	x_1
y-coordinate of P	y_1	y_1	$f(x_1)$
x-coordinate of Q	x_2	$x_1 + \Delta x$	$x_1 + \Delta x$
y-coordinate of Q	y_2	$y_1 + \Delta y$	$f(x_1 + \Delta x)$
change in x	$x_2 - x_1$	Δx	Δx
change in y	$y_2 - y_1$	Δy	$f(x_1 + \Delta x) - f(x_1)$

Note: In Example 4 in order to find an expression for $f(x_1 + \Delta x)$, we have to expand the cube of a binomial:
$(x_1 + \Delta x)^3 = x_1^3 + 3x_1^2\Delta x + 3x_1(\Delta x_1)^2 + (\Delta x_1)^3$.

Note: Figure 3-19 shows that the line tangent to the curve at (.5, 2.125) intersects the curve at another point in the second quadrant.

In Example 5 (page 81) it is pointed out that m_{PQ} is an **average rate of change** whereas $m_{\tan}$ is an **instantaneous rate of change.**

Exercises: page 81: 1, 3, 5, 7, 9, 11, 15, 17, 25.